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THE GENERAL THEORY OF CYCLIC-HARMONIC CURVES.

By Robert E. Moritz.

1. Introduction.

1.1. *Cyclic-harmonic motion* may appropriately be defined as motion resulting from the composition of simple-harmonic motion in a straight line with uniform rotatory motion about a fixed point in this line. The locus of the resultant motion is a *cyclic-harmonic curve.*

1.2. Let the simple-harmonic motion be represented by the equation

$$\rho = a \cos pt + k,$$

where a is the amplitude of the vibration, $2\pi/p$ the period, k the distance of the mean point of vibration from the origin of coördinates and t the time. The rotatory motion of a line about the origin may be represented by the equation

$$\theta = qt,$$

where q is the rate of rotation. On eliminating t from these two equations we obtain

$$\rho = a \cos \frac{p}{q}\theta + k, \tag{1}$$

the equation of the cyclic-harmonic curve expressed in polar coördinates.

1.3. The foregoing derivation of equation (1) suggests the following convenient method of constructing by points any cyclic-harmonic curve whose equation is given.

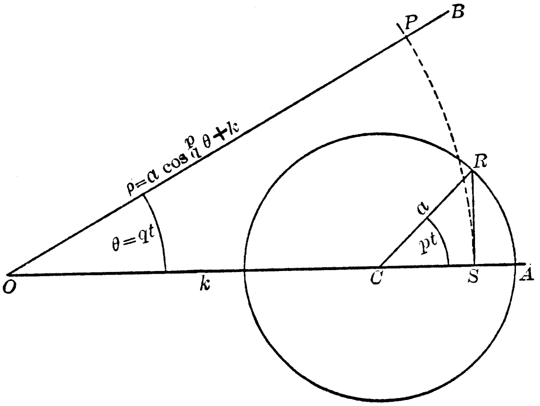

Fig. 1.

Let OA (Fig. 1) represent the initial line, O the pole. From O on OA lay off OC equal to k and with C as a center and a radius equal to a describe the circle of reference of the simple-harmonic motion. Select any convenient unit of angular measure and construct angles ACR and AOB equal to pt and qt units respectively, t being any arbitrarily chosen integer. From R draw RS perpendicular to OA and from O as a center and OS as a radius describe the arc SP cutting OB at P. Then P is a point on the cyclic-harmonic curve $\rho = a \cos (p/q)\theta + k$, for

$$\rho = OP = OS = OC + CS = k + a \cos pt = a \cos \frac{p}{q}\theta + k,$$

since $\theta = qt$.

1.4. By choosing the angular unit sufficiently small, and taking in turn $t = 0, 1, 2, 3$, etc., as many points may be constructed as desired and at intervals small at will. Figures 2, 3, 4, 5 show the method applied to the construction of the cyclic-harmonics $\rho = a \cos 2\theta + k$, for the values $k = 3a$, $k = a$, $k = a/3$, $k = 0$, respectively. Corresponding points on the circle of reference and the cyclic-harmonic are numbered alike.

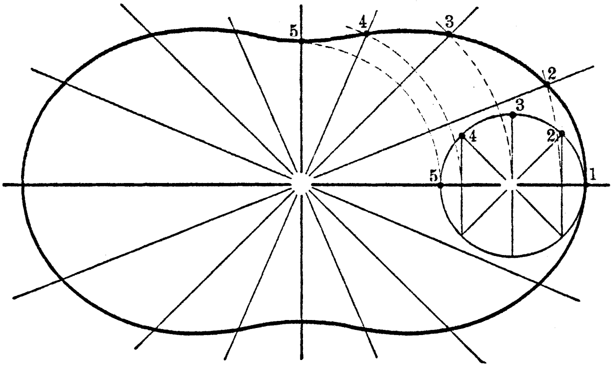

FIG. 2.

1.5. An inspection of the preceding figures discloses certain properties which are independent of the particular value of the ratio p/q employed and which are therefore common to all the species of the genus determined by p/q. Figure 2 has an open center and it follows from the mode of construction, as is otherwise obvious from the form of the equation, that the curve is confined between two circles whose radii are $k - a$ and $k + a$ respectively. Figure 3 consists of leaves which meet in cusps at the origin. The axial diameter of these leaves is $k + a$. Figure 4 consists of two sets of leaves with bases meeting at the origin. The axial diameter of one set of leaves is $k - a$, of the other $k + a$. Figure 5 consists of a single whorl of equal leaves whose axial diameter is a.

1.6. These properties serve as a convenient basis for the classification of the cyclic-harmonics of a given genus p/q. We shall call a cyclic-harmonic curve *curtate* if $k > a$, *cuspitate* if $k = a$, *prolate* if $k < a < 0$, *equi-foliate* if $k = 0$.

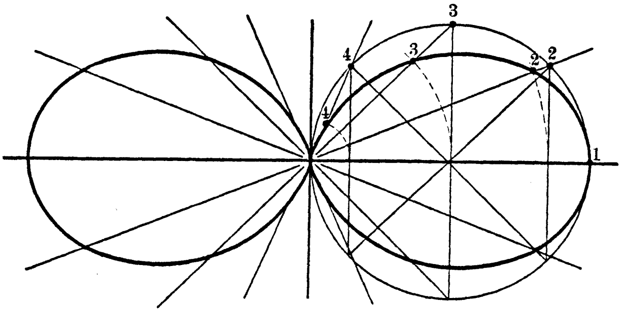

FIG. 3.

1.7. We have assumed the phase of the harmonic motion equal to zero. This restriction may be removed by writing the equation of the harmonic motion in the form

$$\rho = a \cos (pt - \epsilon) + k,$$

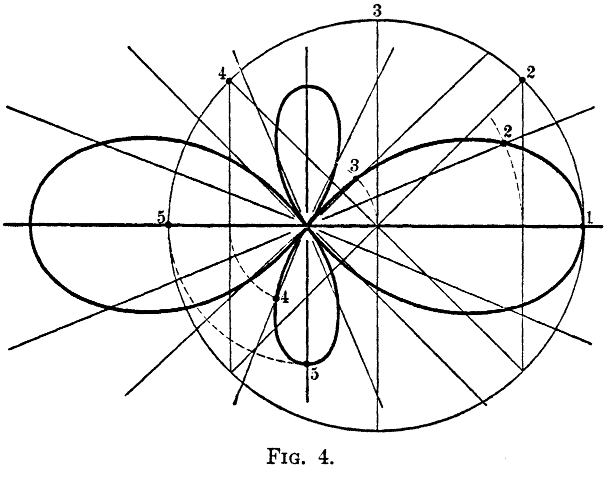

FIG. 4.

where ϵ/p is the phase of the vibration; the equation of the resultant motion then becomes

$$(2) \qquad \rho = a \cos \left(\frac{p}{q} \theta - \epsilon \right) + k.$$

Equation (2) is only apparently more general than equation (1) for the former goes over into the latter if we put $\theta = \theta' + \epsilon q/p$, that is, if we turn the initial line through an angle $\epsilon q/p$. There is, therefore, no loss in generality in the curves to be considered if we assume the phase of the vibration equal to zero.

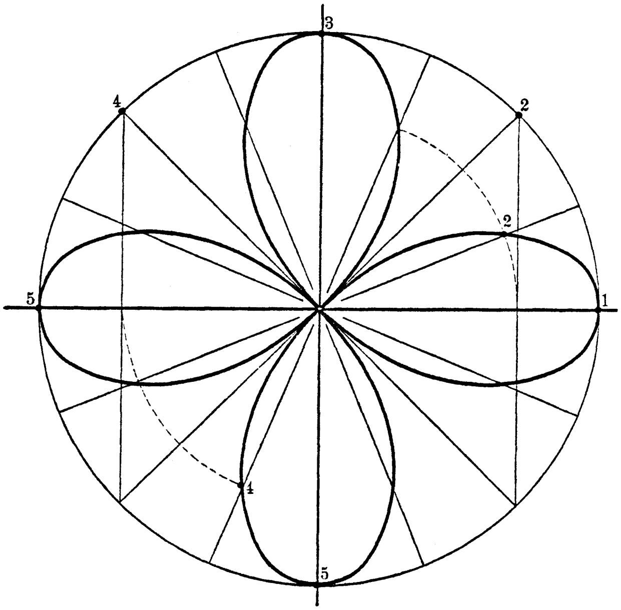

FIG. 5.

1.8. Furthermore, in studying the properties of cyclic-harmonic curves, we may assume a and k both positive. For if a is negative, $a = -a'$,

$$(3) \quad \rho = a \cos \frac{p}{q}\theta + k = -a' \cos \frac{p}{q}\theta + k = a' \cos\left(\frac{p}{q}\theta + \pi\right) + k;$$

if k is negative, $k = -k'$, then

$$(4) \quad \rho = a \cos \frac{p}{q}\theta + k = a \cos \frac{p}{q}\theta - k'$$
$$= -\left[a \cos\left(\frac{p}{q}\theta + \pi\right) + k'\right];$$

and if a and k are both negative, $a = -a'$, $k = -k'$, then

$$(5) \qquad \rho = a \cos \frac{p}{q}\theta + k = -\left(a' \cos \frac{p}{q}\theta + k'\right).$$

Now (3) is of the form (2), and (4) and (5) differ from (2) and (1) respectively only in the sign of ρ, that is, the curves represented by (3), (4), and (5) differ from those represented by (1) in position only. In studying the properties of the curves represented by the equation

$$(6) \qquad \rho = a \cos \left(\frac{p}{q} \theta + \epsilon \right) + k, \qquad a,\ k,\ \epsilon \text{ positive or negative,}$$

we may therefore, without loss in generality, assume a and k positive and ϵ equal to zero.

1.9. Cyclic-harmonic curves are algebraic or transcendental according as p/q is rational or irrational. For since the curve lies entirely within the circle of radius $a + k$, it will cut a straight line in a finite or infinite number of points according as it does or does not return into itself, that is, according as it is, or is not, possible to satisfy the equation

$$\frac{\rho - k}{a} = \cos \frac{p}{q} \theta = \cos \frac{p}{q} (\theta + 2n\pi) , \qquad n \text{ integral.}$$

This equation is satisfied only provided

$$\frac{p}{q} (\theta + 2n\pi) = \frac{p}{q} \theta + 2m\pi, \qquad m \text{ integral,}$$

that is, provided

$$\frac{p}{q} = \frac{m}{n}, \qquad \text{a rational fraction.}$$

We shall throughout the remainder of this paper impose the restriction that p/q be rational and, unless otherwise stated, shall use the term cyclic-harmonic subject to this restriction.

1.10. Cyclic-harmonic curves, as is apparent from their equation, embrace a considerable number of well-known curves. Notable among these are the cardioids and Pascal's limaçon,* Freeth's nephroid,† Münger's double egg curve,‡ and the roses or foliate curves.§ But the simple mode of generation, which gives rise to all of these curves and an infinite number of others, seems to have escaped the observation of previous investigators, there appears not even a record of a common class name or of any attempt at classification. So likewise the many beautiful properties common to all of these curves appear never to have been brought to light, for the reason, no doubt, that in the special cases, which have been carefully studied, these properties are so veiled as to elude detection. It is only from the larger point of view and for the larger values of p and q that these general properties appear in their real significance.

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* Roberval, Observations sur la composition des mouvements, Mem. de l'Acad. Royal des Sci., VI, Paris, 1730.

† Proc. Lond. Math. Soc., vol. 10 (1879).

‡ Die eiförmigen Kurven; Dissertation, Bern, 1894.

§ Grandi, Flores geometrici, etc., Florence, 1728. Auth. Dissertation, Marburg, 1866. Hyde, Foliate Curves, The Analyst, II, 1875. Himstedt, Progr. Löbau, 1888.

2. Cyclic-harmonics in Cartesian Coördinates.

2.1. We assume that p/q is positive, rational, and reduced to its lowest terms, so that p is relatively prime to q. Furthermore, unless the contrary is explicitly stated, we shall assume that $k \neq 0$.* The cyclic-harmonic, $\rho = a\cos(p/q)\theta + k$, may then be rationally expressed in terms of Cartesian coördinates, x, y, as follows:

Consider the identity

$$(\cos\theta + i\sin\theta)^p = \cos p\theta + i\sin p\theta = \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q \cdot$$

If we expand the first member of this identity in terms of powers of $\cos\theta$ and $\sin\theta$ and the last member in terms of powers of $\cos(p/q)\theta$ and $\sin(p/q)\theta$ and then equate the real parts of the two expansions we obtain the new identity

$$\sum_{n=0}^{i}[(-1)^n C_{2n}^{p}\cos^{p-2n}\theta\sin^{2n}\theta] = \sum_{m=0}^{j}\left[(-1)^m C_{2m}^{q}\cos^{q-2m}\left(\frac{p}{q}\theta\right)\sin^{2m}\left(\frac{p}{q}\theta\right)\right],$$

where i and j represent the integral parts of $p/2$ and $q/2$ respectively.

Now $\cos\theta = x/\rho$, $\sin\theta = y/\rho$, and from the equation of the cyclic-harmonic curve we have

$$\cos\frac{p}{q}\theta = (\rho - k)/a, \qquad \sin\frac{p}{q}\theta = [a^2 - (\rho - k)^2]^{1/2}/a.$$

On substituting these values in the foregoing identity and multiplying through by the factor $a^q\rho^p$ to avoid fractions, we obtain

$$a^q\sum_{n=0}^{i}[(-1)^n C_{2n}^{p}x^{p-2n}y^{2n}] = \rho^p\sum_{m=0}^{j}[(-1)^m C_{2m}^{q}(\rho - k)^{q-2m}\{a^2 - (\rho - k)^2\}^m], \tag{7}$$

a rational equation between x, y, and ρ.

2.2. The left member of equation (7) is a polynomial, homogeneous of degree p, in x and y. The right member is a polynomial of degree $p + q$ in ρ, consisting of ρ^p multiplied into a polynomial of degree q in $(\rho - k)$. The coefficients of the successive terms $(\rho - k)^q$, $(\rho - k)^{q-2}$, $(\rho - k)^{q-4}$, $\cdots$, $(\rho - k)^{q-2m}$, are

$$B_q, \; -a^2B_{q-2}, \; +a^4B_{q-4}, \; \cdots, \; (-1)^m a^{2m}B_{q-2m},$$

where

$$B_q = 1 + C_2^q + C_4^q + C_6^q + \cdots,$$

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* It will be shown that $k = 0$ gives rise to degenerate forms for which many of the general theorems here deduced break down. Nor is it necessary to consider these special cases at length since their properties have been repeatedly investigated. See Loria, Spezielle algebraische und transscendente ebene Kurven, Leipzig (1902), Absch. 5, Kap. 8.

$$\begin{aligned}
B_{q-2} &= C_2{}^q + C_1{}^2C_4{}^q + C_2{}^3C_6{}^q + C_3{}^4C_8{}^q + \cdots,\\
B_{q-4} &= C_4{}^q + C_1{}^3C_6{}^q + C_2{}^4C_8{}^q + C_3{}^5C_{10}{}^q + \cdots,\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ,\\
B_{q-2m} &= C_{2m}{}^q + C_1{}^{m+1}C_{2m+2}{}^q + C_2{}^{m+2}C_{2m+4}{}^q + C_3{}^{m+3}C_{2m+6}{}^q + \cdots.
\end{aligned}$$

It is now easy to write out the coefficients of the various powers of ρ in the expanded form of the right member of equation (7). Denoting the coefficient of ρ^{p+r} by C_r we have

$$\begin{aligned}
C_q &= B_q,\\
C_{q-1} &= -\cdot kC_1{}^qB_q,\\
C_{q-2} &= k^2C_2{}^qB_q - a^2B_{q-2},\\
C_{q-3} &= -\ k^3C_3{}^qB_q + ka^2C_1{}^{q-2}B_{q-2},\\
&\qquad\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ,\\
C_{q-r} &= k^rC_r{}^qB_q - k^{r-2}a^2C_{r-2}{}^{q-2}B_{q-2} + k^{r-4}a^4C_{r-4}{}^{q-4}B_{q-4} - \cdots\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (-1)^{r/2}a^rB_{q-r},\\
&= -\ [k^rC_r{}^qB_q - k^{r-2}a^2C_{r-2}{}^{q-2}B_{q-2} + k^{r-4}a^4C_{r-4}{}^{q-4}B_{q-4} - \cdots\\
&\qquad\qquad\qquad\qquad + (-1)^{(r-1)/2}ka^{r-1}C_1{}^{q-r+1}B_{q-r+1}]
\end{aligned}$$

according as r is even or odd. We note in particular that

$$C_q = [(1+1)^q + (1-1)^q]/2 = 2^{q-1}, \qquad C_{q-1} = -\ kq\cdot 2^{q-1}, \tag{8}$$

$$\begin{aligned}
C_0 &= \pm\ [k^qB_q - k^{q-2}a^2B_{q-2} + k^{q-4}a^4B_{q-4} - \cdots\\
&= \frac{\pm\ [(k + \sqrt{k^2 - a^2})^q + (k - \sqrt{k^2 - a^2})^q]}{2}.
\end{aligned} \tag{9}$$

2.3. It is obvious that the coefficients of the terms in ρ of the right member of equation (7) are independent of p, that is, these coefficients are invariant for all cyclic-harmonics having the same value of q. Similarly, the coefficients of the terms in x and y of the left member are, barring the factor a^q, independent of q, hence these coefficients are invariant for all values of q. This property greatly facilitates the computation of the Cartesian equations of the various genera of cyclic-harmonics. It should further be observed that the computation of the B's which enter C's may be expedited by superimposing one on the other the two arrays

$$\begin{array}{cccccc}
C_0{}^q & C_2{}^q & C_4{}^q & C_6{}^q & C_8{}^q & \cdot\\
C_2{}^q & C_4{}^q & C_6{}^q & C_8{}^q & C_{10}{}^q & \cdot\\
C_4{}^q & C_6{}^q & C_8{}^q & C_{10}{}^q & C_{12}{}^q & \cdot\\
C_6{}^q & C_8{}^q & C_{10}{}^q & C_{12}{}^q & C_{14}{}^q & \cdot\\
C_8{}^q & C_{10}{}^q & C_{12}{}^q & C_{14}{}^q & C_{16}{}^q & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot &
\end{array}
\qquad
\begin{array}{cccccc}
C_0{}^0 & C_1{}^1 & C_2{}^2 & C_3{}^3 & C_4{}^4 & \cdot\\
C_0{}^1 & C_1{}^2 & C_2{}^3 & C_3{}^4 & C_4{}^5 & \cdot\\
C_0{}^2 & C_1{}^3 & C_2{}^4 & C_3{}^5 & C_4{}^6 & \cdot\\
C_0{}^3 & C_1{}^4 & C_2{}^5 & C_3{}^6 & C_4{}^7 & \cdot\\
C_0{}^4 & C_1{}^5 & C_2{}^6 & C_3{}^7 & C_4{}^8 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Any required value B_{q-2m} may then be obtained by adding the products of the superimposed terms in the $(n + 1)$th row or column.

2.4. The following table contains the coefficients B_{q-2m} for all values of q from 1 to 10 inclusive.

B_{q-2m}

m \ q	1	2	3	4	5	6	7	8	9	10
0	1	2	4	8	16	32	64	128	256	512
1		1	3	8	20	48	112	256	576	1280
2				1	5	18	56	160	432	1120
3						1	7	32	120	400
4								1	9	50
5										1

2.5. The foregoing table forms the basis for the computation of the coefficients C_{q-r} which are tabulated below.

C_{q-r}

r \ q	1	2	3	4	5	6	7	8	9	10	
0	1	2	4	8	16	32	64	128	256	512	
1	1	4	12	32	80	192	448	1024	2304	5120	$\times - k$
2		2	12	48	160	480	1344	3584	9216	23040	$\times k^2$
		1	3	8	20	48	112	256	576	1280	$\times - a^2$
3			4	32	160	640	2240	7168	21504	61440	$\times - k^3$
			3	16	60	192	560	1536	4032	10240	$\times + ka^2$
4				8	80	480	2240	8960	32256	107520	$\times k^4$
				8	60	288	1120	3840	12096	35840	$\times - k^2a^2$
				1	5	18	56	160	432	1120	$\times + a^4$
5					16	192	1344	7168	32256	129024	$\times - k^5$
					20	192	1120	5120	20160	71680	$\times k^3a^2$
					5	36	168	640	2160	6720	$\times - ka^4$
6						32	448	3584	21504	107520	$\times k^6$
						48	560	3840	20160	89600	$\times - k^4a^2$
						18	168	960	4320	16800	$\times + k^2a^4$
						1	7	32	120	400	$\times - a^6$
7							64	1024	9216	61440	$\times - k^7$
							112	1536	12096	71680	$\times + k^5a^2$
							56	640	4320	22400	$\times - k^3a^4$
							7	64	360	1600	$\times + ka^5$
8								128	2304	23040	$\times k^8$
								256	4032	35840	$\times - k^6a^2$
								160	2160	16800	$\times + k^4a^4$
								32	360	2400	$\times - k^2a^6$
								1	9	50	$\times + a^8$
9									256	5120	$\times - k^9$
									576	10240	$\times + k^7a^2$
									432	6720	$\times - k^5a^4$
									120	1600	$\times + k^3a^6$
									9	100	$\times - ka^8$
10										512	$\times k^{10}$
										1280	$\times - k^8a^2$
										1120	$\times + k^6a^4$
										400	$\times - k^4a^6$
										50	$\times + k^2a^8$
										1	$\times - a^{10}$

2.6. The foregoing table of coefficients enables us to write the Cartesian equation of each of the 63 genera of cyclic-harmonic curves, p, $q \leqq 10$, in decreasing powers of ρ. The degree of the equation in ρ is $p + q$ for by (8) the coefficient ρ^{p+q} is different from zero. Now $\rho^2 = x^2 + y^2$, hence the equation is not rational in x and y unless only even powers of ρ occur, but it appears from (8) that for $k \neq 0$ at least one odd power of ρ is present in the equation. One quadrature is therefore necessary and sufficient to rationalize the equation.

2.7. The process of rationalization is exceedingly laborious for all but the lower values of q. It required several days intense work on the part of the writer to compute the coefficients in the Cartesian equation whose equivalent polar form is $\rho = a \cos (\theta/10) + k$. Written at length this equation is:

$$\begin{aligned}
&262{,}144(x^2+y^2)^{11}-(2{,}621{,}440k^2+1{,}310{,}720a^2)(x^2+y^2)^{10}+(11{,}796{,}480k^4\\
&+9{,}175{,}040k^2a^2+2{,}785{,}280a^4)(x^2+y^2)^9-(31{,}457{,}280k^6+26{,}214{,}400k^4a^2\\
&+13{,}107{,}200k^2a^4+3{,}276{,}800a^6)(x^2+y^2)^8+(55{,}050{,}240k^8+36{,}700{,}160k^6a^2\\
&+22{,}937{,}600k^4a^4+9{,}830{,}400k^2a^6+2{,}329{,}600a^8)(x^2+y^2)^7-(66{,}060{,}288k^{10}\\
&+18{,}350{,}080k^8a^2+18{,}350{,}080k^6a^4+9{,}830{,}400k^4a^6+4{,}147{,}200k^2a^8\\
&+1{,}025{,}024a^{10})(x^2+y^2)^6+(55{,}050{,}240k^{12}-18{,}350{,}080k^{10}a^2+11{,}468{,}800k^8a^4\\
&+3{,}276{,}800k^6a^6+2{,}176{,}000k^4a^8+977{,}920k^2a^{10}+274{,}560a^{12})(x^2+y^2)^5\\
&-(31{,}457{,}280k^{14}-36{,}700{,}160k^{12}a^2+18{,}350{,}080k^{10}a^4-3{,}276{,}800k^8a^6\\
&+716{,}800k^6a^8+215{,}040k^4a^{10}+120{,}320k^2a^{12}+42{,}240a^{14})(x^2+y^2)^4\\
&+(11{,}796{,}480k^{16}-26{,}214{,}400k^{14}a^2+22{,}937{,}600k^{12}a^4-9{,}830{,}400k^{10}a^6\\
&+2{,}176{,}000k^8a^8-215{,}040k^6a^{10}+19{,}200k^4a^{12}+6{,}400k^2a^{14}+3{,}300a^{16})(x^2+y^2)^3\\
&-(2{,}621{,}440k^{18}-9{,}175{,}040k^{16}a^2+13{,}107{,}200k^{14}a^4-9{,}830{,}400k^{12}a^6\\
&+4{,}147{,}200k^{10}a^8-977{,}920k^8a^{10}+120{,}320k^6a^{12}-6{,}400k^4a^{14}+200k^2a^{16}\\
&+100a^{18})(x^2+y^2)^2+(262{,}144k^{20}-1{,}310{,}720k^{18}a^2+2{,}785{,}280k^{16}a^4\\
&-3{,}276{,}800k^{14}a^6+2{,}329{,}600k^{12}a^8-1{,}025{,}024k^{10}a^{10}+274{,}560k^8a^{12}\\
&-42{,}240k^6a^{14}+3{,}300k^4a^{16}-100k^2a^{18}+a^{20})(x^2+y^2)-10{,}240ka^{10}x(x^2+y^2)^5\\
&-(122{,}880k^2-20{,}480a^2)ka^{10}x(x^2+y^2)^4-(258{,}048k^4-143{,}360k^2a^2\\
&+13{,}440a^4)ka^{10}x(x^2+y^2)^3-(122{,}880k^6-143{,}360k^4a^2+44{,}800k^2a^4\\
&-3{,}200a^6)ka^{10}x(x^2+y^2)^2-(10{,}240k^8-20{,}480k^6a^2+13{,}440k^4a^4-3{,}200k^2a^6\\
&+200a^8)ka^{10}x(x^2+y^2)-a^{20}x^2=0.
\end{aligned}$$

2.8. We have seen that every cyclic-harmonic curve, $k \neq 0$, leads to an algebraic equation of degree $p + q$ in the ρ's which requires one quadrature in order to make the equation rational in x and y, therefore,

Every cyclic-harmonic curve, $\rho = a \cos (p/q)\theta + k$, $k \neq 0$, is an algebraic curve of order $2(p + q)$.

2.9. If $k = 0$ the second member of (7) reduces to

$$\rho^p\Sigma[(-1)^m C_{2m}\rho^{q-2m}(a^2 - \rho^2)^m],$$

whose degree in ρ is odd or even according as $p + q$ is odd or even. If $p + q$ is even, that is, if both p and q are odd, the equation is rational in x and y as it stands, hence

Cyclic-harmonic curves for which both p and q are odd, and $k = 0$, are algebraic curves of order $p + q$.

Every cyclic-harmonic curve is, therefore, an algebraic curve of even order.

2.10. By considering the rationalized form of the equation of the general cyclic-harmonic curve we see that when both p and q are odd and $k = 0$ this equation reduces to the product of two equal factors. The cyclic-harmonic curve corresponding to this case consists of two equal and coincident branches each of order $p + q$. We shall call a single branch of such a curve a *degenerate* cyclic-harmonic curve. It is obvious that all results derived for the general case will require modification before they can be applied to the degenerate case. Unlike degenerate forms of many other plane curves, degenerate cyclic-harmonic curves cannot be readily recognized by their form alone.

2.11. The number of genera of cyclic-harmonic curves of a given order is readily determined as follows. Let $2n$ be the given order, then the number in question is evidently the number of integral solutions, p, q, of the equation $p + q = n$ subject to the restriction that p, q and n be relatively prime to each other. It follows that any pair of integers, p and $n - p$, of which p is less than n and relatively prime to n, constitute a solution, for if p is relatively prime to n so also is $n - p$. The number sought is, therefore, the *totient function* $\varphi(n)$, and we have the theorem, *The number of genera of cyclic-harmonic curves, $k \neq 0$, having a given order $2n$ is the totient number $\varphi(n)$.*

The foregoing enumeration does not include degenerate cyclic-harmonics. Of such there are $\varphi(2n)$ having a given order $2n$.

The number of genera of cyclic-harmonic curves, $k \neq 0$, whose order is $2n$ or less is $\sum_{n=1}^{n}\varphi(n)$. Besides these there are $\sum_{n=1}^{n}\varphi(2n)$ degenerate forms of order $2n$ or lower order.

2.12. Another interesting inquiry concerns the number of genera of cyclic-harmonic curves for which neither p nor q exceeds a given number n. Suppose first $p > q$, then for a given value of p there are $\varphi(p)$ admissible values of q and hence of the ratio p/q. Now p may take all values from 1 to n inclusive, hence there are $\sum_{n=1}^{n}\varphi(n)$ genera subject to the restriction $n \geqq p > q$. Evidently there is an equal number subject to the restriction $n \geqq q > p$. Finally there is the case $p = q = 1$. Hence

The number of genera of cyclic-harmonic curves, $k \neq 0$, subject to the condition that neither p nor q shall exceed a given number n is $2\sum_{n=1}^{n}\varphi(n) + 1$.

2.13. Returning to equation (7) we see that the order of the terms of lowest degree in x and y is p, after rationalization $2p$, hence

Every cyclic-harmonic curve, $k \neq 0$, *has a multiple point of order* $2p$ *at the origin.*

Every degenerate cyclic-harmonic curve has a multiple point of order p *at the origin.*

2.14. The terms of highest order in the rationalized form of equation (7) result from the expansion of

$$\rho^{2(p+q)} = (x^2 + y^2)^{p+q} = (x + iy)^{p+q}(x - iy)^{p+q};$$

each of the circular rays $x + iy = 0$, $x - iy = 0$, must therefore intersect the curve in $p + q$ coincident points on the line at infinity, hence

Every cyclic-harmonic curve, $k \neq 0$, *has* $p + q - 1$ *fold contact with the line at infinity at each of the circular points.*

The line at infinity is thus seen to be a double tangent to every cyclic-harmonic curve; it is an ordinary or inflectional tangent according as $p + q$ is even or odd; moreover since the order of the curve is $2(p + q)$ it can meet the line at infinity in no points other than the circular points.

2.15. That cyclic-harmonic curves are unipartite is sufficiently obvious from their definition; that they are also unicursal or rational may be shown as follows. Since $\rho = a \cos (p/q)\theta + k$,

$$x = \left(a \cos \frac{p}{q} \theta + k\right) \cos \theta, \qquad y = \left(a \cos \frac{p}{q} \theta + k\right) \sin \theta.$$

Now let $\theta/q = \varphi$, then $\theta = q\varphi$, and we have

$$x = (a \cos p\varphi + k) \cos q\varphi, \qquad y = (a \cos p\varphi + k) \sin q\varphi.$$

Now p and q being integers, $\cos p\varphi$ $\cos q\varphi$ and $\sin q\varphi$, may each be rationally expressed in terms of $\sin \varphi$ and $\cos \varphi$ so that on making the substitution

$$\cos \varphi = (1 - t^2)/(1 + t^2), \qquad \sin \varphi = 2t/(1 + t^2), \qquad t = \tan (\varphi/2),$$

x and y may each be expressed rationally in terms of the single parameter t, hence

All cyclic-harmonic curves are rational or unicursal curves.